Hierarchical Competition in Femtocell-Based Cellular Networks

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***Abstract*—This paper considers the downlink power allocation problem in a cellular network where a bi-level hierarchy exists. The network is comprised of the macrocells underlaid with femtocells. The objective of each station in the network is to maximize its capacity under power constraints. The problem is formulated as a Stackelberg game with the macrocell base stations as the leaders and the femtocell access points as the followers. The leaders are assumed to have enough information and foresight to consider the response of the followers while formulating their strategies. To characterize such interaction between leaders and followers, Stackelberg equilibrium is introduced; and it is shown to exist under the assumption of continuity of best response function of the leader sub-game.**

Keywords: Femtocell, cellular network, game theory, Stackelberg equilibrium.

I. Introduction

Imperfect network coverage, especially in the interiors of houses and buildings, is one of the problems of existing cellular system. The traditional way of solving this problem is to deploy more base stations (BSs) so as to increase the network coverage. However, this approach is economically inefficient. A new concept has recently emerged to overlay *low power* and *low cost* base stations, called femto-access points (FAP), on the existing cellular network technology (2G, 3G, WiMAX). This femto-access point is connected with IP backhaul through a local broadband connection, such as DSL, cable, or fiber. FAP will be a low cost, simple plug-and-play device, just like WiFi-AP. In [1], [2], [3] a general survey of this technology is given. In [4] uplink capacity and interference avoidance for two-tier femtocell networks is considered. [5] considers downlink power control in femtocell overlay where the objective is to minimize the transmit power under signal-to-noise constraint. [6] deals with power allocation along similar lines.

Various benefits of using FAPs have been identified. The first advantage is that FAPs improve indoor coverage where macrocell base station signal can be weak. Also, FAPs provide high data rate and improved quality-of-services to the subscribers, at the same time, it lengthens the battery life of the mobile phones since the mobile phones do not need to communicate with a distant BS. For the operators, it saves the backhaul cost since FAP traffic is carried over wired residential broadband connections. Finally, FAPs point a way towards the convergence of landline and mobile services.

In this paper, it is assumed that the deployment of FAPs over the existing cellular system creates a hierarchical overlay network. We address the issue of transmit power control by each station in the downlink scenario. Since FAPs are expected to operate on the same frequency band as macro BSs, co-channel interference can impede the overall performance of the network. As the number of FAPs increases, the accumulated interference becomes a critical issue.

Here, another interesting situation can also be envisaged when a mobile is being connected to a FAP, while another cell-edge user is being connected to a BS. In this case there is a strong interference from the BS to the mobile within the coverage of the FAP. Please note that the cell-edge user is outside the coverage of the FAP, so it can be served only by the BS, which requires higher transmit power. Therefore, some form of power control is required to avoid performance degradation to the mobile terminals served by macro BSs.

We model the power control problem as a Stackelberg game, or the leader-follower game, that allows a distributed power allocation algorithm to be implemented. The BSs are considered to be the leaders, whereas the FAPs are considered to be the followers in the leader-follower game. The game is divided into two sub-games such that the leaders and followers not only compete against the other group, but each player also competes with other players of their own group. The solution of such a game is the Stackelberg equilibrium, which is an extended form of the Nash equilibrium. If no hierarchy is present, the Stackelberg game would reduce to the Nash game, which is the simple non-cooperative game, and the solution would be the Nash equilibrium. In this paper, we show the existence of such Stackelberg equilibrium. Lastly, a low-complexity algorithm is adopted to obtain the Stackelberg equilibrium; and its performance compared with waterfilling methods.

II. System Model

Consider a system of transceivers made up of a number of base station transceivers and femto cell access points. Let the set of base station transceivers be given by $\mathcal{M} = \{1, \dots, M\}$, and the set of femtocell access points be given by $\mathcal{N} = \{1, \dots, N\}$, where $\mathcal{M}$ and $\mathcal{N}$ are two disjoint sets. We further assume that there exists a wire line back haul connecting the femtocell access points to the base station transceivers, which enables them to exchange relevant information.

We assume that OFDMA is employed by both the base stations and the femtocell access points. The available spectrum is divided into a set of orthogonal channels $\mathcal{L}_i = \{1, \ldots, L_i\}$ by each transceiver $i \in \mathcal{M} \cup \mathcal{N}$. Each channel has the same bandwidth W_i. For any $i, j \in \mathcal{M} \cup \mathcal{N}$, $k \in \mathcal{L}_i$, $l \in \mathcal{L}_j$, let G_{ij}^{kl} be the link gain for channel l of transmitter j to channel k of receiver i. We assume that the channels are slow, flat fading channels. The transmitters estimate the channel gain based on the feedback obtained from the receivers, and channel state does not change until the next time slot. The orthogonality of the channels of user i assures that the link gain $G_{ii}^{kl} = 0$ for $k \neq l$. Note that the orthogonality of the channels does not need to be retained for different users.

The noise is assumed to be additive, white Gaussian, with power n_i^k in channel k of receiver i. Let the transmit power of transmitter j in channel l be denoted by p_j^l. The interference and noise power as observed by receiver i in channel k is given by

$$\nu_i^k = \sum_{j \neq i} \sum_{l \in \mathcal{L}_j} G_{ij}^{kl} p_j^l + n_i^k. \tag{1}$$

Normalizing the link gains and noise powers such that $G_{ii}^{kk} = 1$ for all $i \in \mathcal{M} \cup \mathcal{N}$ and $k \in \mathcal{L}_i$, the signal-to-interference-plus-noise power (SINR) of receiver i in channel k, which equals $G_{ii}^{kk} p_i^k / \nu_i^k$, can be simplified to be

$$\frac{p_i^k}{\nu_i^k} = \frac{p_i^k}{\sum_{j \neq i} \sum_{l \in \mathcal{L}_j} G_{ij}^{kl} p_j^l + n_i^k}. \tag{2}$$

If all the users divide the spectrum in the same fashion, that is, the number of channels are identical, and the overlapping channels share the same spectrum[1], then we say that the channel is a *parallel Gaussian interference channel*. For parallel Gaussian interference channel, we have $L_i = L_j$ for every $i, j \in \mathcal{M} \cup \mathcal{N}$, and the link gain $G_{ij}^{kl} = 0$ whenever $k \neq l$. Hence, the SINR of receiver i at channel k is simplified to

$$\frac{p_i^k}{\nu_i^k} = \frac{p_i^k}{\sum_{j \neq i} G_{ij}^{kk} p_j^k + n_i^k}. \tag{3}$$

Let the $\boldsymbol{p}_i = (p_i^1, \ldots, p_i^{L_i})^T$ be the transmit *power vector* of transmitter i and $\boldsymbol{\nu}_i = (\nu_i^1, \ldots, \nu_i^{L_i})^T$ be the *interference (plus noise) vector* of receiver i. The power of each user i is subject to the total power constraint such that $\sum_{k \in \mathcal{L}_i} p_i^k \leq \bar{p}_i$, and individual power constraint (also known as *spectral mask*) $p_i^k \leq \bar{m}_i^k$ for all $i \in \mathcal{N} \cup \mathcal{M}$ and $k \in \mathcal{L}_i$. We assume that $\bar{p}_i \leq \sum_{k \in \mathcal{L}_i} \bar{m}_i^k$, for all i, so as to avoid trivial cases. If this inequality does not hold, then $p_i^k = \bar{m}_i^k$ for all i and k. Let $\mathcal{P}_i$ denote the set of all feasible power vectors of transmitter i as follows:

$$\mathcal{P}_i = \Big\{ \boldsymbol{p}_i \in \prod_{k \in \mathcal{L}_i} [0, \bar{m}_i^k] : \sum_{k \in \mathcal{L}_i} p_i^k \leq \bar{p}_i \Big\}. \tag{4}$$

We assume that the base stations and femtocell access points seek to allocate their transmit power to maximize the total throughput. Given the power allocation vector, from Shannon's capacity formula for additive white Gaussian channels, the maximal data rate that user i can achieve is

$$\begin{aligned} C_i &= C_i(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_i, \ldots, \boldsymbol{p}_{M+N}) = C_i(\boldsymbol{p}_i, \boldsymbol{\nu}_i) \\ &= W_i \sum_{k \in \mathcal{L}_i} \log \Big(1 + \frac{p_i^k}{\nu_i^k}\Big). \end{aligned} \tag{5}$$

III. STACKELBERG GAME FORMULATION

Stackelberg game, also known as the leader-follower game or the bi-level game, is an extension of non-cooperative game in which there is a group of players, called leaders, that have the privilege of making the first move, while the remaining players, called the followers, make their move after the leaders. Therefore, a distinct hierarchy exists among the players; and the leaders can anticipate, and take into consideration, the behavior of the followers, before making their own moves. The followers do not have this power to anticipate the leaders' move.

A. Stackelberg Equilibrium

Here we consider a game of complete and perfect information. Let a bi-level hierarchy exist among the set of transceivers of aforementioned system. The transceivers belonging to the upper level are referred to as leaders, whereas the transceivers belonging to the lower level are referred to as followers. The leaders are the macrocell base station transceivers $\mathcal{M}$, while the followers are the set of femtocell access points $\mathcal{N}$. Therefore, the total set of players in the Stackelberg game is $\mathcal{M} \cup \mathcal{N}$.

The strategy space of the leaders is given by $\mathcal{P}^{up} = \prod_{i \in \mathcal{M}} \mathcal{P}_i$, and the point in $\mathcal{P}^{up}$ is called a leader strategy. The leaders compete with each other in a non-cooperative manner in order to maximize their individual throughput, all the time anticipating the response of the followers. This sub-game is referred to as the upper sub-game, and its equilibrium, the upper sub-game equilibrium. After the leaders apply their strategies, the followers make their moves in response to the leaders' strategies. The strategy space of the followers is $\mathcal{P}^{low} = \prod_{i \in \mathcal{N}} \mathcal{P}_i$, and a point in $\mathcal{P}^{low}$ is called a follower strategy. The followers also compete with each other in a non-cooperative manner[2] to maximize their own throughput; and this sub-game is referred to as the lower sub-game, and its equilibrium, the lower sub-game equilibrium. Lastly, the strategy space of the entire game is given by the Cartesian product $\mathcal{P} = \mathcal{P}^{up} \times \mathcal{P}^{low}$.

Let us denote a best response function by

$$\begin{aligned} \boldsymbol{p}_i &= \operatorname*{argmax}_{\boldsymbol{p}_i} C_i(\boldsymbol{p}_i, \boldsymbol{p}_{-i}) \\ &= BR_i(\boldsymbol{p}_{-i}; \bar{p}_i, \bar{\boldsymbol{m}}_i) \end{aligned} \tag{6}$$

where $\bar{\boldsymbol{m}}_i = (\bar{m}_i^k)_{k \in \mathcal{L}_i}$ such that it maximizes the ith user's capacity function subject to the power constraints. Also $-i$ denote all the users in the set $\mathcal{M} \cup \mathcal{N}$ except user i.

[1]Here, we assume that the frequency reuse is one among users in different cells, along with uniform and random frequency-hopping patterns, where full overlapping is allowed.

[2]This can be referred to as the "noisy neighbors" problem.

We define the lower sub-game equilibrium as any fixed point $\boldsymbol{p}^{low^*} = (\boldsymbol{p}_1^*, \ldots, \boldsymbol{p}_N^*) \in \mathcal{P}^{low}$ such that

$$\boldsymbol{p}_i^* = BR_i(\boldsymbol{p}_{-i}^*, \boldsymbol{p}^{up}; \bar{p}_i, \bar{\boldsymbol{m}}_i) \quad (7)$$

where $\boldsymbol{p}^{up} \in \mathcal{P}^{up}$ is *fixed* but *arbitrary* leader strategy, for all $i \in \mathcal{N}$. Note that this definition is the same as the Nash equilibrium of the lower game. Since every user in the lower sub-game will myopically maximize their individual throughput, the best response $BR_i(\cdot)$ of each user in the sub-game will be given by the waterfilling function from (21) in Appendix. Defining $BR^{low} \equiv (BR_i(\cdot))_{i=1}^N$, we can express the lower sub-game equilibrium as any fixed point of the system power space $\boldsymbol{p}^* \in \mathcal{P}$ such that

$$\boldsymbol{p}^* = BR^{low}(\boldsymbol{p}^*). \quad (8)$$

We now define the upper sub-game equilibrium as any fixed point $\boldsymbol{p}^{up*} = (\boldsymbol{p}_1^*, \ldots, \boldsymbol{p}_M^*) \in \mathcal{P}^{up}$ such that

$$\boldsymbol{p}_i^* = BR_i(\boldsymbol{p}_{-i}^*, \boldsymbol{p}^{low^*}; \bar{p}_i, \bar{\boldsymbol{m}}_i) \quad (9)$$

where $\boldsymbol{p}^{low^*} \in \mathcal{P}^{low}$ is an *equilibrium* follower strategy conditioned on the upper sub-game strategy, for all $i \in \mathcal{M}$. Equivalently, let $BR^{up} \equiv (BR_i(\cdot))_{i=1}^M$, then we can define the upper sub-game equilibrium as the fixed point $\boldsymbol{p}^{up*} \in \mathcal{P}^{up}$ such that

$$\boldsymbol{p}^{up*} = BR^{up}(\boldsymbol{p}^{up*}; BR^{low}(\boldsymbol{p}^{low^*}; \boldsymbol{p}^{up*})). \quad (10)$$

We can simplify the notations further and write the upper sub-game equilibrium in terms of system power vector as any fixed point $\boldsymbol{p}^* \in \mathcal{P}$ such that

$$\boldsymbol{p}^* = BR^{up}(BR^{low}(\boldsymbol{p}^*)). \quad (11)$$

Note that although the function $BR^{up}(\cdot)$ acts only on the upper sub-game strategy, the lower sub-game equilibrium strategy associated with each upper sub-game strategy needs to be computed as well.

The Stackelberg equilibrium is defined as any fixed point $(\boldsymbol{p}^{up*}, \boldsymbol{p}^{low^*}) = \boldsymbol{p}^* \in \mathcal{P}$, that satisfies (8) and (11). In other words, let the self map $BR : \mathcal{P} \to \mathcal{P}$ be a composition of the two vector functions

$$BR \equiv BR^{up} \circ BR^{low}.$$

Then, we have the Stackelberg equilibrium as any fixed point of the function BR,

$$\boldsymbol{p}^* = BR(\boldsymbol{p}^*) \quad (12)$$

such that $\boldsymbol{p}^* = (\boldsymbol{p}^{up*}, \boldsymbol{p}^{low^*})$. This definition of Stackelberg equilibrium is the same as sub-game perfect Nash equilibrium, which is a refinement of the Nash equilibrium for dynamic games.

B. Existence of Stackelberg Equilibrium

It is important to note that the best response function for the lower sub-game $BR^{low}(\cdot) \equiv (F_i(\cdot))_{i=1}^N$, where $F_i(\cdot)$ is given by the waterfilling function from (21), is a piecewise affine continuous function of $\boldsymbol{p}$ [7]. Assuming the continuity of the best response function of the upper sub-game BR^{up}, we can prove the existence of a Stackelberg equilibrium using Schauder fixed point theorem. For convenience, the Schauder fixed point theorem is stated below:

Schauder fixed point theorem: Every continuous function from a convex compact subset $\mathcal{K}$ of a Banach space to $\mathcal{K}$ itself has a fixed point [8].

Therefore, we have the following theorem:

Theorem 1: Assuming the best response function of the upper sub-game is continuous, at least one Stackelberg equilibrium exists for the leader-follower power allocation game.

Proof: Since the best response functions $BR^{up} : \mathcal{P} \to \mathcal{P}$ and $BR^{low} : \mathcal{P} \to \mathcal{P}$ are continuous functions, the composition of these two functions $BR = BR^{up} \circ BR^{low}$ is also continuous. It is also easy to see that $\mathcal{P}$ as defined by (15) is convex, closed, and bounded. Since Stackelberg equilibrium is defined as any fixed point of $BR(\cdot)$, from Schauder fixed point theorem, the above statement is proved. ■

Corollary 1: At least one upper sub-game and one lower sub-game equilibrium exist.

Indeed, it can be shown that the lower sub-game equilibrium is in fact unique. It is straightforward to check that, for any fixed leader strategy in $\mathcal{P}^{up}$, the sum of capacities $\sum_{i \in \mathcal{N}} C_i$ for the lower game is diagonally strictly concave. Therefore, from [9], for any given leader strategy, there is at most one lower sub-game equilibrium. However, the uniqueness of the upper sub-game equilibrium cannot be guaranteed.

If, however, the leaders also adopt their strategies according to the waterfilling function, assuming the interference vector to be constant, then the Stackelberg equilibrium will be the same as the Nash equilibrium of the whole game in static form. In such a case, the uniqueness of Nash equilibrium can be shown by noting that the upper sub-game would then be a concave game as well, thus guaranteeing the uniqueness of the Nash equilibrium.

IV. Algorithms and Numerical Results

A. Algorithm Implementation

The direct implementation of waterfilling algorithm to iteratively locate the fixed point of the lower sub-game usually gives an unstable system for a random channel gain matrix. Furthermore the system is stable only when the waterfilling function is a contraction [7]. Therefore, the following Mann iterative technique, which allows a weaker stability criterion, is used to ensure that stability can be achieved:

$$\boldsymbol{p}_i(t+1) = (1 - \lambda(t))\boldsymbol{p}_i(t) + \lambda(t)F_i(\boldsymbol{p}_{-i}(t)) \quad (13)$$

such that the scalar sequence $\{\lambda(t)\}$ satisfies

1) $\lambda(t = 0) = 1$,

2) $\lambda(t) \in (0,1)$ for $t > 0$, and
3) $\sum_{t=0}^{\infty} \lambda(t) = \infty$ [10].

It can be assumed for the $\mathcal{P}^{low}$ to be a uniformly convex Banach space and $F(\cdot)$ to be a quasi nonexpansive operator on $\mathcal{P}^{low}$ for the iteration to reach the lower sub-game equilibrium provided that $\{\lambda(t)\}$ is bounded away from 0 and 1 (Theorem 4.5 in [10]). The $\{\lambda(t)\}$ sequence is generated as

$$\lambda(t) = \frac{t}{2t+1}, \quad t > 0. \tag{14}$$

It is straightforward to verify that the sequence satisfies the constraints on $\lambda(t)$ and as $\lim_{t\to\infty} \lambda(t) = 1/2$.

On the other hand, a simple method to compute the upper sub-game equilibrium does not exist, in part because every time the leaders' strategy is updated, the lower sub-game equilibrium need to be computed. Such a problem is intrinsically difficult to solve and a number of algorithms have been proposed [11]. In Algorithm 1 of [11], the authors give a Lagrangian dual approach for computing the best response of a leader. The idea is to approximate the Lagrangian dual function by locally optimizing the Lagrangian with respect to individual frequency bin while keeping the power in other bins constant, for a fixed dual variable. It then updates the dual variable by using the bi-section search and repeats the procedure until convergence is achieved.

However, the algorithm is very sensitive to initial starting point and the ordering of iterations. Also, since it does not perform an exhaustive search, it is, at best, suboptimal since it does not decouple the the power allocated in each frequency bin from the interference generated in the other frequency bins to compute the Lagrangian dual function.

To find the Nash equilibrium, where both the leaders and followers use the waterfilling function as their best response function, the upper sub-game equilibrium is first computed, assuming an initial lower sub-game equilibrium. The lower sub-game equilibrium is then computed while keeping the upper sub-game equilibrium fixed. The upper sub-game and the lower sub-game equilibrium is iteratively computed until the system power vector of the whole game converges. As mentioned earlier, Mann iterative method is used to obtain the sub-game equilibriums.

B. Parameter Setting

For the purpose of numerical simulations, parallel Gaussian interference channel is assumed. The number of base stations (leaders) is chosen to be $M = 2$, and the number of femtocell access points is chosen to be $N = 3$. The number of channel is taken as $L_i = 4$ for all $i \in \mathcal{M} \cup \mathcal{N}$. For the sake of simplicity, spectral mask is assumed to be absent and the noise power vector is taken to be unity in each channel. The total power of every user is taken to be the same, and is referenced against the total noise power in all channels. Assuming independent and identically distributed Rayleigh fading channels, the system channel gain matrix is formed using exponential random variable with parameter 0.5. The sum of direct channel gains g_{ii}^k is normalized to unity, and that of cross channel gains g_{ij}^k is normalized to 0.25. The waterfilling algorithm is implemented using binary search method.

C. Numerical Results

The simulation is performed using random instances of channel gain matrix. Fig. 1 shows the average capacity per user, as measured in bps/Hz, versus SNR per user, as measured in dB. It is seen that, on an average, there is an overall deterioration of system performance at the Stackelberg equilibrium, as implemented using the Lagrangian dual method in [11], when compared with the performance of the Nash equilibrium. The result is not so surprising when we consider the fact that the Lagrangian dual method is at best a suboptimal approach.

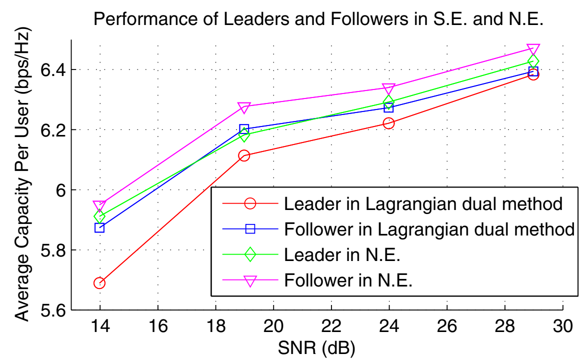

Fig. 1. Comparison between the performance of a leader and a follower.

V. Conclusion

This paper addresses the issue of allocating transmit power over cellular networks comprising of macrocells underlaid with femtocells. The users' objective is to maximize their individual capacity. But unlike femtocell access points, the macrocell base stations are assumed to have enough information to predict the response of the femtocells for given macrocell power profile. Therefore, a Stackelberg game is formulated with the macrocell base stations as the leaders and the femtocell access points as the followers. The game is divided into two sub-games, each sub-game comprising of the set of leaders, referred to as the upper sub-game, and the set of followers, referred to as the lower sub-game. The candidates in each sub-game compete with each other in a non-cooperative manner to reach a sub-game Nash equilibrium. A Stackelberg equilibrium is shown to exist. Due to the notorious computational burden of estimating the Stackelberg equilibrium, a low complexity algorithm based on Lagrangian dual theory [11] is chosen. However the numerical results show that the adopted algorithm is suboptimal.

Appendix A
Waterfilling Procedure

Let the *system power vector* set is given by the Cartesian product

$$\mathcal{P} = \prod_{i \in \mathcal{M} \cup \mathcal{N}} \mathcal{P}_i. \tag{15}$$

The optimal method of allocating transmit power over different channels of user i so as to maximize its data rate, for any given power vector of its opponents $-i$, is given by the "waterfilling" procedure. It is also known as the myopic response in the sense that it neglects the possible evolution of the competitors' strategies, and instead considers their strategy to be fixed. Although (5) is a non-convex function, assuming that the opponents' power allocation remains constant makes the equation convex with respect to $\boldsymbol{p}_i$.

Theorem 2: Given that $\boldsymbol{p}_{-i} \in \mathcal{P}_{-i}$ remains fixed for user $i \in \mathcal{M} \cup \mathcal{N}$, the power vector that maximizes $C_i(\boldsymbol{p}_i, \boldsymbol{\nu}_i)$ over all power vectors in $\mathcal{P}_i$ if and only if there is a "water-level" $\omega \geq 0$ so that

$$p_i^k = \begin{cases} 0 & \text{if } \omega - \nu_i^k \leq 0 \\ \bar{m}_i^k & \text{if } \omega - \nu_i^k \geq \bar{m}_i^k \\ \omega - \nu_i^k & \text{otherwise,} \end{cases} \tag{16}$$

for all $k \in \mathcal{L}_i$, and $\sum_{k \in \mathcal{L}_i} p_i^k = \bar{p}_i$.

We can prove this by using the method of Lagrange multipliers. This theorem appears as is given in [7]. Similar theorems are given in [12], [13]. The value of ω in (16) has to be determined numerically.

Based on the waterfilling procedure, for any user $i \in \mathcal{M} \cup \mathcal{N}$, we can partition the available set of channels $\mathcal{L}_i$ into three disjoint sets. For any $k \in \mathcal{L}_i$, if the optimal power in the kth channel is non-zero, the kth channel is referred to as an *active* channel. If the associated power of the kth channel equals to its upper bound $\bar{m}_i^k$, it is referred to as being *saturated*. Therefore, let $\mathcal{S}_i$ be the set of all saturated channels, and $\mathcal{A}_i$ be the set of all channels that are active but not saturated. Then, based on the knowledge of these sets, we can rewrite (16) as follows:

$$p_i^k = \begin{cases} 0, & \text{for } k \in \mathcal{L}_i \setminus \mathcal{A}_i \cup \mathcal{S}_i \\ \bar{m}_i^k, & \text{for } k \in \mathcal{S}_i \\ \omega - \nu_i^k, & k \in \mathcal{A}_i. \end{cases} \tag{17}$$

Substituting the expression for optimal power (16) in the total power constraint $\sum_{k \in \mathcal{A}_i \cup \mathcal{S}_i} p_i^k = \bar{p}_i$, we can obtain

$$\omega = \frac{1}{|\mathcal{A}_i|} \big(\bar{p}_i + \sum_{k \in \mathcal{A}_i} \nu_i^k - \sum_{k \in \mathcal{S}_i} \bar{m}_i^k \big) \tag{18}$$

where $|\mathcal{A}_i|$ is the cardinality of set $\mathcal{A}_i$. From (16) and (18), we have for $k \in \mathcal{A}_i$

$$p_i^k = \Big(-1 + \frac{1}{|\mathcal{A}_i|} \Big) \nu_i^k + \frac{1}{|\mathcal{A}_i|} \sum_{\substack{l \in \mathcal{A}_i \\ l \neq k}} \nu_i^l + \frac{1}{|\mathcal{A}_i|} \big(\bar{p}_i - \sum_{k \in \mathcal{S}_i} \bar{m}_i^k \big). \tag{19}$$

Therefore, we see that the sets $\mathcal{A}_i$ and $\mathcal{S}_i$ are sufficient to completely characterize a waterfilling solution. Using this expression, the waterfilling equation as given in (16) has been expressed in matrix form by [7] as follows,

$$\begin{aligned} \boldsymbol{p}_i &= F(\boldsymbol{p}_{-i}; \bar{p}_i, \bar{\boldsymbol{m}}_i) \qquad (20) \\ &= \boldsymbol{W}_i(\mathcal{A}_i) \boldsymbol{\nu}_i + \boldsymbol{b}_i(\mathcal{A}_i, \mathcal{S}_i), \end{aligned}$$

where $\boldsymbol{W}_i(\mathcal{A}_i)$ is an $L_i \times L_i$ symmetric matrix whose (k,l)-th member is given by

$$[\boldsymbol{W}_i(\mathcal{A}_i)]_{kl} = \begin{cases} 0 & \text{if } k \text{ or } l \notin \mathcal{A}_i \\ \frac{1}{|\mathcal{A}_i|} & \text{if } k, l \in \mathcal{A}_i \text{ and } k \neq l \\ -1 + \frac{1}{|\mathcal{A}_i|} & \text{if } k, l \in \mathcal{A}_i \text{ and } k = l; \end{cases}$$

and $\boldsymbol{b}_i(\mathcal{A}_i, \mathcal{S}_i)$ is an L_i dimensional column vector given by

$$b_i^k(\mathcal{A}_i, \mathcal{S}_i) = \begin{cases} 0, & \text{if } k \notin \mathcal{A}_i \cup \mathcal{S}_i \\ \bar{m}_i^k, & \text{if } k \in \mathcal{S}_i \\ \frac{1}{|\mathcal{A}_i|}(\bar{p}_i - \sum_{l \in \mathcal{S}_i} \bar{m}_i^k), & \text{if } k \in \mathcal{A}_i. \end{cases}$$

It should again be noted that the elements of sets $\mathcal{A}_i$ and $\mathcal{S}_i$ are determined using some algorithm.

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